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# On Smoothness of the Green Function for the Complement of a Rarefied Cantor-Type Set

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**Abstract** Smoothness of the Green functions for the complement of rarefied Cantor-type sets is described in terms of the function  $\varphi(\delta) = (1/\log \frac{1}{\delta})$  that gives the logarithmic measure of sets. Markov's constants of the corresponding sets are evaluated.

**Keywords** Green's function · Markov's inequality · Cantor-type sets

**Mathematics Subject Classification (2000)** 31A15 · 41A10 · 41A17

## 1 Introduction

Let a compact set  $K$  be regular with respect to the Dirichlet problem. Then the Green function  $g_{\mathbb{C} \setminus K}$  of  $\mathbb{C} \setminus K$  with pole at infinity is continuous throughout  $\mathbb{C}$ . Related to polynomial inequalities and some other applications, the problem of smoothness of  $g_{\mathbb{C} \setminus K}$  near the boundary of  $K$  has attracted the attention of many mathematicians (see, e.g., the survey [4] and the references given there). New incentive to analyze the problem has been provided by the monograph [17] by Totik, where the author characterized the smoothness of Green functions and harmonic measures in terms of the density  $\Theta_K(t)$  (Theorems 2.1 and 2.2 in [17]). For the case  $K \subset [0, 1]$  with  $0 \in K$ , which we will consider in what follows, the density at 0 is measured by the function  $\Theta_K(t) = m([0, t] \setminus K)$ , where  $m$  stands for the linear Lebesgue measure.

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The monotonicity of the Green function with respect to the set  $K$  implies  $g_{\mathbb{C} \setminus K}(z) \geq g_{\mathbb{C} \setminus [0,1]}(z)$  for  $z \in \mathbb{C}$ . In this way, we get the optimal behavior ( $Lip_{\frac{1}{2}}$  smoothness) near the origin of the function  $g_{\mathbb{C} \setminus K}$  for  $K \subset [0, 1]$ . Various conditions for optimal smoothness of  $g_{\mathbb{C} \setminus K}$  in terms of metric properties of the set  $K$  are suggested in [9, 17] and in papers of V. Andrievskii [2–4]. For example, the Green function corresponding to the classical Cantor set  $K_0$  is Hölder continuous by [6], but is not optimal smooth, by Theorem 5.1 in [17]. A recent result on smoothness of  $g_{\mathbb{C} \setminus K_0}$  can be found in [15].

Here we consider Cantor-type sets  $K^{(\alpha)}$  with “lowest smoothness” of the corresponding Green function. Let  $1 < \alpha$ ,  $0 < l_1 < \frac{1}{2}$ , and  $2l_1^{\alpha-1} < 1$ . Then  $K^{(\alpha)} = \bigcap_{s=0}^{\infty} E_s$ , where  $E_0 = I_{1,0} = [0, 1]$ ,  $E_s$  is a union of  $2^s$  closed *basic* intervals  $I_{j,s}$  of length  $l_s = l_{s-1}^{\alpha}$ , and  $E_{s+1}$  is obtained by deleting the open concentric subinterval of length  $h_s := l_s - 2l_{s+1}$  from each  $I_{j,s}$  with  $j = 1, 2, \dots, 2^s$ . The set  $K^{(\alpha)}$  is not polar if and only if  $\alpha < 2$  ([8, Chap. IV, Theorem 3]). Also, by Pleśniak [13], in the case of the Cantor type set, the corresponding set is regular if and only if it is not polar. Thus, in the case  $1 < \alpha < 2$ , the Green function  $g_{\mathbb{C} \setminus K^{(\alpha)}}$  is continuous. We show that its modulus of continuity can be estimated in terms of the function  $\varphi(\delta) = (1/\log \frac{1}{\delta})$ , which is used in the definition of the logarithmic measure (see, e.g., [12, Chap. V, 6]). Here and subsequently,  $\log$  denotes the natural logarithm.

Since  $\Theta_{K^{(\alpha)}}(t) = t$ , neither the estimation from Theorem 2.2 in [17] nor the previous general bound of Green functions given by Tsuji [18, Theorem III, 67] can be applied to our case. Let  $\Pi_n$  denote the set of all polynomials of degree at most  $n$ ,  $\Pi = \bigcup_{n=0}^{\infty} \Pi_n$ . Let  $|f|_K := \sup_{x \in K} |f(x)|$ . We use the representation

$$g_{\mathbb{C} \setminus K^{(\alpha)}}(z) = \sup \left\{ \frac{\log |P(z)|}{\deg P} : P \in \Pi, \deg P \geq 1, |P|_{K^{(\alpha)}} \leq 1 \right\}, \quad (1)$$

which follows on one hand from the Bernstein–Walsh lemma ([19, p. 77]) and on the other hand by the possibility of approximating  $\exp g_{\mathbb{C} \setminus K^{(\alpha)}}(z)$ , for example by the sequence  $(|\Phi_n(z)|^{1/n})$ , where  $\Phi_n$  denotes the normalized Fekete polynomial (see, e.g., [14, Theorem 11.1]).

There is a strong connection between the smoothness of  $g_{\mathbb{C} \setminus K}$  near the boundary of  $K$  and values of Markov’s factors  $M_n(K) = \sup_{P \in \Pi_n} \frac{|P'|_K}{|P|_K}$ , which are well defined for any infinite set  $K$ . Indeed, suppose that for some increasing continuous function  $F$  we have  $g_{\mathbb{C} \setminus K}(z) \leq F(\delta)$  for  $\text{dist}(z, K) \leq \delta$ . Then for any  $P \in \Pi_n$  the Bernstein–Walsh inequality gives  $|P(z)| \leq |P|_K \exp[n \cdot F(\delta)]$ . Applying Cauchy’s formula for  $P'$  on the circle with center at  $\zeta \in K$  and of radius  $\delta$  yields  $|P'(\zeta)| \leq \delta^{-1} \exp[n \cdot F(\delta)] |P|_K$ . This gives the bound  $M_n(K) \leq \inf_{\delta} \delta^{-1} \exp[n \cdot F(\delta)]$ . Particularly, if we choose  $\delta$  with  $F(\delta) = n^{-1}$ , then  $M_n(K) \leq e \cdot [F_{-1}(n^{-1})]^{-1}$ , where  $F_{-1}$  stands for the inverse to  $F$  function. For example, the Hölder continuity of the Green function  $g_{\mathbb{C} \setminus K}$  implies Markov’s property of the set  $K$ , which means that there are constants  $C, r$  such that  $M_n(K) \leq Cn^r$  for all  $n$ .

Here we give an asymptotic for  $M_n(K^{(\alpha)})$  which is new compared to the previous results about Markov’s constants of Cantor-type sets (see [10, Example 7], [16], and [7]).

As a method we employ local interpolations of functions that were used in [1] to present extension operators for the Whitney spaces  $\mathcal{E}(K^{(\alpha)})$  and in [11] to construct topological bases in spaces  $\mathcal{E}(K)$  for more general Cantor-type sets.

## 2 Results

Given  $1 < \alpha < 2$ , let  $K^{(\alpha)}$  be the Cantor set defined in the introduction,  $\varphi(\delta) = (\log \frac{1}{\delta})^{-1}$  for  $0 < \delta < 1$  and  $\gamma = \log \frac{2}{\alpha} / \log \alpha$ .

**Theorem 1** *For every  $0 < \varepsilon < \gamma$  there exist constants  $\delta_0, C_0$ , depending on  $\alpha$  and  $\varepsilon$ , such that  $g_{\mathbb{C} \setminus K^{(\alpha)}}(z) \leq C_0 \varphi^{\gamma-\varepsilon}(\delta)$  for  $z \in \mathbb{C}$  with  $\text{dist}(z, K^{(\alpha)}) = \delta \leq \delta_0$ .*

**Theorem 2** *There are constants  $\delta_0, \varepsilon_0$ , depending only on  $\alpha$ , such that  $g_{\mathbb{C} \setminus K^{(\alpha)}}(-\delta) \geq \varepsilon_0 \varphi^\gamma(\delta)$  for  $\delta \leq \delta_0$ .*

**Corollary 1** *If  $1 < \alpha < 2$ , then for every  $0 < \varepsilon < \gamma$  there exists a constant  $C_1$  such that  $M_n(K^{(\alpha)}) \leq \exp[C_1 \cdot n^{(1+\varepsilon) \frac{\log \alpha}{\log 2}}]$  for  $n \in \mathbb{N}$ . On the other hand, for each  $\alpha > 1$  we have  $M_n(K^{(\alpha)}) > \exp[\alpha^{-2} \cdot n^{\frac{\log \alpha}{\log 2}}]$  for  $n \in \mathbb{N}$ .*

## 3 Proof of Theorem 2

Let us first prove the more simple sharpness result.

Without loss of generality we can suppose that  $l_1 = e^{-1}$ , so  $l_s = \exp(-\alpha^{s-1})$ . If  $l_{q+1} < \delta \leq l_q$ , then  $\alpha^{-q} < \varphi(\delta) \leq \alpha^{-q+1}$ . Since  $\alpha^{-\gamma} = \alpha/2$ , we have

$$\left(\frac{\alpha}{2}\right)^q < \varphi^\gamma(\delta) \leq \left(\frac{\alpha}{2}\right)^{q-1}. \quad (2)$$

Let us fix  $q_0$  with  $(\alpha/2)^{q_0-1} \leq (\alpha-1)/2$ ,  $\delta_0 = l_{q_0}$ , and  $\varepsilon_0 = \frac{\alpha}{8} \frac{\alpha-1}{2-\alpha}$ . In view of (1) and (2), it is enough for given  $l_{q+1} < \delta \leq l_q \leq l_{q_0}$  to find a polynomial  $P \in \Pi_n$  with  $|P|_{K^{(\alpha)}} \leq 1$  such that

$$\frac{\log |P(-\delta)|}{n} \geq \frac{1}{4} \frac{\alpha-1}{2-\alpha} \left(\frac{\alpha}{2}\right)^q. \quad (3)$$

For fixed  $m \in \mathbb{N}$  let  $(x_k)_{k=1}^{2^m}$  be the set of all endpoints of the basic intervals  $I_{j,m-1}$  with  $j = 1, 2, \dots, 2^{m-1}$ . We arrange them in increasing order, so  $x_1 = 0, x_2 = l_{m-1}, x_3 = l_{m-2} - l_{m-1}, \dots, x_{2^k} = l_{m-k}, \dots, x_{2^m} = 1$ . Set  $\omega(z) = \prod_{k=1}^{2^m} (z - x_k)$ . Then the fundamental Lagrange polynomial  $L_1(z) = (\omega(z)/z \cdot \omega'(0))$  has the norm  $|L_1|_{K^{(\alpha)}}$  equal 1, as is easy to check. Indeed, if  $x \in K^{(\alpha)} \cap I_{1,m-1}$ , then  $|L_1(x)| \leq |L_1(0)| = 1$ , by the monotonicity of  $\omega(z)/z$  there. Otherwise,  $x \in K^{(\alpha)} \cap I_{j,m-1}$  with  $j = 2, \dots, 2^{m-1}$ , and  $|\omega(x)| \leq l_m \cdot x_2 \cdot x_3 \cdots x_{2^m}$ , so  $|L_1(x)| \leq l_m/x < 1$ .

Now for given  $q$  we take  $m = 2q$ ,  $n = 4^q - 1$  and  $P = L_1 \in \Pi_n$ . Then

$$|P(-\delta)| = \prod_{k=2}^{2^{q-1}} \frac{x_k + \delta}{x_k} \prod_{k=2^{q-1}+1}^{4^q} \left(1 + \frac{\delta}{x_k}\right).$$

We disregard the second product, which exceeds 1, and  $x_k$  in the numerator of the first product. For its denominator we have  $\prod_{k=2}^{2^{q-1}} x_k < l_{2q-1} \cdot l_{2q-2}^2 \cdots l_{q+1}^{2^{q-2}} = l_{q+1}^{\varkappa}$ , where  $\varkappa = 2^{q-2} + \alpha \cdot 2^{q-3} + \cdots + \alpha^{q-3} \cdot 2 + \alpha^{q-2} = (2-\alpha)^{-1} (2^{q-1} - \alpha^{q-1})$ . Therefore,  $|P(-\delta)| > l_{q+1}^{2^{q-1}-1-\varkappa} = \exp[(\varkappa - 2^{q-1} + 1)\alpha^q]$ . Here,  $\varkappa - 2^{q-1} = 2^{q-1}[\frac{\alpha-1}{2-\alpha} - \frac{1}{2-\alpha}(\frac{\alpha}{2})^{q-1}] > 2^{q-2}\frac{\alpha-1}{2-\alpha}$ , due to the choice of  $q_0$ . Thus,  $\log |P(-\delta)| > 2^{q-2}\alpha^q\frac{\alpha-1}{2-\alpha}$ . This gives the desired bound (3), since  $n < 4^q$ .  $\square$

#### 4 Proof of Theorem 1

Let us fix  $\varepsilon$  with  $0 < \varepsilon < \gamma$ . As above, we suppose that  $l_1 = e^{-1}$ .

We want to find  $q_0$  and  $C_0$  such that if  $\text{dist}(z, K^{(\alpha)}) = \delta \in (l_{q+1}, l_q]$  with  $q \geq q_0$ , then

$$g_{\mathbb{C} \setminus K^{(\alpha)}}(z) \leq C_0 \left(\frac{\alpha}{2}\right)^q \alpha^{q\varepsilon}. \quad (4)$$

We set  $Q_\alpha := \frac{\alpha}{\alpha-1} \log \frac{2}{\alpha}$  and choose  $q_0$  so large that for  $q \geq q_0$  the following conditions hold:

$$Q_\alpha q < \alpha^{q-1}, \quad (5)$$

$$\log(Q_\alpha q) < q\varepsilon \log^2 \alpha, \quad (6)$$

$$\left(\frac{\alpha}{2}\right)^q < \frac{1}{4}. \quad (7)$$

Now for fixed  $q \geq q_0$  we take  $k = [\frac{\log(Q_\alpha q)}{\log \alpha}] + 1$ , where  $[x]$  denotes the greatest integer in  $x$ . Due to the choice of  $Q_\alpha$ , we have

$$l_k^{\alpha-1} < \left(\frac{\alpha}{2}\right)^q \leq l_{k-1}^{\alpha-1}, \quad (8)$$

that is,  $k = \min\{j : l_j^{\alpha-1} < (\frac{\alpha}{2})^q\}$ . Since (5) is equivalent to  $l_{q-1}^{\alpha-1} < (\frac{\alpha}{2})^q$ , we get  $k < q$ . Also (6) implies

$$2^k < 2\alpha^{q\varepsilon}, \quad (9)$$

as is easy to check.

Arguing as in the proof of Theorem 2.2 in [17], we see that it is enough to consider (4) only for  $z = -\delta$ . Let us fix any polynomial  $P$  with  $|P|_{K^{(\alpha)}} \leq 1$ . Let  $m \in \mathbb{N}$  be such that  $2^{m-1} \leq \deg P < 2^m$ . In view of (1), we can reduce (4) to

$$\log |P(-\delta)| \leq C_0 2^{m-q} \alpha^{q(1+\varepsilon)}, \quad (10)$$

and what is more, since polynomials in the representation (1) can be of arbitrary large degree, we can suppose without loss of generality that  $m \geq 2q$ .

We interpolate  $P$  on the interval  $I_{1,k}$  at  $2^m$  endpoints of  $I_{j,k+m-1}$  with  $j = 1, 2, \dots, 2^{m-1}$ . Thus,  $x_1 = 0, x_2 = l_{k+m-1}, x_3 = l_{k+m-2} - l_{k+m-1}, \dots, x_{2^i} = l_{k+m-i}, \dots, x_{2^m} = l_k$ . Here  $\omega(z) = \prod_{i=1}^{2^m} (z - x_i)$  and  $L_j(z) = \frac{\omega(z)}{(z - x_j) \cdot \omega'(x_j)}$  for  $1 \leq j \leq 2^m$ .

Since  $\deg P < 2^m$ , the interpolating polynomial  $\mathcal{L}_{2^m-1} = \sum_{j=1}^{2^m} P(x_j) L_j$  coincides with  $P$ . In our case  $|P(x_j)| \leq 1$ . Therefore,

$$|P(-\delta)| \leq \sum_{j=1}^{2^m} |L_j(-\delta)|.$$

Let us fix any  $1 \leq j \leq 2^m$  and estimate  $|L_j(-\delta)|$  from above. We have  $|\omega(-\delta)| < l_q \cdot (l_q + l_{k+m-1}) \cdot (l_q + l_{k+m-2})^2 \cdots (l_q + l_k)^{2^{m-1}} = l_q^{2^{k+m-q}} \cdot 2^{2^{k+m-q-1}} \cdot l_{q-1}^{2^{k+m-q}} \cdots l_k^{2^{m-1}} \cdot B$ , where  $B = (1 + \frac{l_{k+m-1}}{l_q})(1 + \frac{l_{k+m-2}}{l_q})^2 \cdots (1 + \frac{l_{q+1}}{l_q})^{2^{k+m-q-2}} (1 + \frac{l_q}{l_{q-1}})^{2^{k+m-q}} \cdots (1 + \frac{l_q}{l_k})^{2^{m-1}}$ . On the other hand, by the structure of the set  $K^{(\alpha)}$ ,  $|\omega'(x_j)| > l_{k+m-1} \cdot h_{k+m-2}^2 \cdot h_{k+m-3}^4 \cdots h_k^{2^{m-1}} = l_{k+m-1} \cdot l_{k+m-2}^2 \cdots l_k^{2^{m-1}} \cdot \beta$  with  $\beta = (1 - 2^{\frac{l_{k+m-1}}{l_{k+m-2}}})^2 \cdots (1 - 2^{\frac{l_{k+1}}{l_k}})^{2^{m-1}}$ . Also,  $|\delta - x_j| \geq l_{q+1}$ . Therefore,

$$|L_j(-\delta)| \leq \frac{B}{\beta} 2^{2^{k+m-q-1}} l_q^{\varkappa},$$

with  $\varkappa = 2^{k+m-q-1} - \alpha - [\alpha^{k+m-q-1} + 2\alpha^{k+m-q-2} + \cdots + 2^{k+m-q-2}\alpha] = 2^{k+m-q-1} - \alpha - 2^{k+m-q-1} \cdot \frac{\alpha}{2-\alpha} + \frac{\alpha^{k+m-q}}{2-\alpha} = -2^{k+m-q} \cdot \frac{\alpha-1}{2-\alpha} + \frac{\alpha^{k+m-q}}{2-\alpha} - \alpha$ .

From this,  $|P(-\delta)| \leq 2^m \frac{B}{\beta} 2^{2^{k+m-q-1}} l_q^{\varkappa}$ , and the desired inequality (10) is analogous to

$$m \log 2 + \log B - \log \beta + 2^{k+m-q-1} \log 2 + \varkappa \log l_q \leq C_0 2^{m-q} \alpha^{q(1+\varepsilon)}.$$

Here,  $\varkappa \log l_q = 2^{k+m-q} \cdot \alpha^{q-1} \cdot \frac{\alpha-1}{2-\alpha} - \frac{\alpha^{k+m-1}}{2-\alpha} + \alpha^q$ . Since  $k+m > 2q$ , the sum of the last two terms is negative. We neglect this sum. Thus it is enough to show

$$m + 2^{k+m-q-1} + \log B - \log \beta + 2^{k+m-q} \cdot \alpha^{q-1} \cdot \frac{\alpha-1}{2-\alpha} \leq C_0 2^{m-q} \alpha^{q(1+\varepsilon)}. \quad (11)$$

Each of the 5 summands on the left will be estimated separately from above by  $R := 2^{m-q} \alpha^{q(1+\varepsilon)}$ .

**S<sub>1</sub>** :=  $m$ . Since  $m \geq 2q$ , we have  $2^{m-q} \geq 2^{m/2} \geq m/2$ , so  $S_1 \leq 2R$ .

**S<sub>2</sub>** :=  $2^{k+m-q-1} < \alpha^{q\varepsilon} 2^{m-q}$ , by (9). Thus,  $S_2 < R$ .

**S<sub>3</sub>** :=  $\log B$ . Clearly,  $l_q/l_{q-1} > l_{q+1}/l_q$ , since  $l_{q-1}^{(\alpha-1)^2} < 1$ . Therefore,  $B < (1 + \frac{l_q}{l_{q-1}})^{2^m}$  and  $\log B < 2^m l_{q-1}^{\alpha-1} < 2^m (\frac{\alpha}{2})^q$ , by (5). Therefore,  $S_3 < R$ .

**S<sub>4</sub>** :=  $-\log \beta$ . Here,  $\frac{l_{k+1}}{l_k} > \frac{l_{j+1}}{l_j}$  for  $j = k+1, \dots, k+m-2$ . Consequently,  $\beta > (1 - 2^{\frac{l_{k+1}}{l_k}})^{2^m}$  and  $\log \beta > -2^{m+2} l_k^{\alpha-1}$ . Here we use the inequality  $\log(1-x) >$

$-2x$ , which is valid for  $0 < x < 1/2$ . In our case  $x = 2l_k^{\alpha-1} < 1/2$ , by (8) and (7). Additionally, (8) implies  $S_4 < 2^{m+2}(\frac{\alpha}{2})^q < 4R$ .

Finally,  $S_5 := 2^{k+m-q} \cdot \alpha^{q-1} \cdot \frac{\alpha-1}{2-\alpha} < \frac{2}{\alpha} \cdot \frac{\alpha-1}{2-\alpha} \cdot R$ , by (9). This gives (11) with  $C_0 = 8 + \frac{2}{\alpha} \cdot \frac{\alpha-1}{2-\alpha}$  and completes the proof of Theorem 1.  $\square$

## 5 Markov's Factors

By the arguments given in the introduction,

$$M_n(K^{(\alpha)}) \leq \inf_{\delta \leq \delta_0} \delta^{-1} \exp[n \cdot C_0 \varphi^{\gamma-\varepsilon}(\delta)].$$

Let us take  $\delta = \exp(-n^{\frac{\log \alpha}{\log 2}})$ . Then  $\varphi(\delta) = n^{-\frac{\log \alpha}{\log 2}}$  and  $n \cdot \varphi^{\gamma}(\delta) = n^{\frac{\log \alpha}{\log 2}}$ . Therefore,  $M_n(K^{(\alpha)}) \leq \exp[(C_0 + 1) \cdot n^{(1+\varepsilon)\frac{\log \alpha}{\log 2}}]$  for large enough  $n$ . By increasing the constant, if necessary, we have the first bound in corollary.

Of course these arguments cannot be used for the case of polar sets  $K^{(\alpha)}$  with  $\alpha \geq 2$ . But the lower bound of  $M_n(K^{(\alpha)})$  can be presented easily for any  $\alpha > 1$ . Indeed, let us fix  $n \in \mathbb{N}$ . Let  $2^m \leq n < 2^{m+1}$ . We take the same  $2^m$  points  $(x_k)_{k=1}^{2^m}$  as in Sect. 3 and  $P(z) = \prod_{k=1}^{2^m} (z - x_k)$ . Then  $|P|_{K^{(\alpha)}} = |P(l_m)| = l_m(x_2 - l_m) \cdots (1 - l_m) < l_m \cdot \prod_{k=2}^{2^m} x_k$ . On the other hand,  $|P'(0)| = \prod_{k=2}^{2^m} x_k$ . By definition, the sequence  $(M_n)$  is not decreasing. Therefore,  $M_n(K^{(\alpha)}) \geq M_{2^m}(K^{(\alpha)}) \geq |P'(0)|/|P|_{K^{(\alpha)}} > l_m^{-1} = \exp \alpha^{m-1} = \exp[\alpha^{-2} \cdot (2^{m+1})^{\frac{\log \alpha}{\log 2}}] > \exp[\alpha^{-2} \cdot n^{\frac{\log \alpha}{\log 2}}]$ .

## 6 Remarks

1. The function  $\varphi(\delta) = (1/\log \frac{1}{\delta})$  was used in [5] to define the *logarithmic dimension* of compact sets, as the Hausdorff dimension corresponding to the function  $\varphi$ . In particular, the logarithmic dimension of  $K^{(\alpha)}$  is  $\frac{\log 2}{\log \alpha}$ .
2. We conjecture that the genuine modulus of continuity of  $g_{\mathbb{C} \setminus K^{(\alpha)}}(z)$  is given by  $\varphi^{\gamma}(\text{dist}(z, K^{(\alpha)}))$ , that is,  $-\varepsilon$  in the upper bound can be removed by another distribution of interpolating nodes, which will be closer to the distribution of the Fekete points on the set  $K^{(\alpha)} \cap I_{1,k}$ . This will mean that  $\exp(\alpha^{-2} \cdot n^{\frac{\log \alpha}{\log 2}}) < M_n(K^{(\alpha)}) < \exp(C \cdot n^{\frac{\log \alpha}{\log 2}})$  for some constant  $C$ .

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